PMM U.S.S.R.,Vol.47,No.3,pp.303-308,1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

UDC 62-50

A NECESSARY OPTIMALITY CONDITION IN A GAME OF ENCOUNTER AT A PRESCRIBED INSTANT*

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A necessary condition is derived, which must be satisfied by the positional strategy minimizing the game's value. It is shown that for a linear system and a convex target function this condition is sufficient. The paper abuts the researches in /1- 4/.

1. We consider a discrete conflict-controlled system described by the equation

$$x[\tau_{i+1}] = f(\tau_i, x[\tau_i]) + u + v, \quad x[\tau_0] = x_0 \tag{1.1}$$
$$u \in P(\tau_i), \quad v \in Q(\tau_i)$$

Here x is an m-dimensional vector, $\tau_0 < \tau_1 < \ldots < \tau_n = \vartheta$ are fixed instants, the function f is continuous together with its partial derivatives with respect to x_j $(j = 1, \ldots, m)$, u and v are the players' controls, $P(\tau_i), Q(\tau_i)$ are convex compacta. Let the target function $\sigma(x)$, continuous together with its partial derivatives, be prescribed. Player u must minimize $\sigma(x)$ at instant $\tau_n = \vartheta$ and player v must maximize it.

The game is formalized as follows. By a positional strategy of its player u we shall mean any function $U = u(\tau_i, x)$ satisfying the inclusion $u(\tau_i, x) \in P(\tau_i)$. Just as in /1,4/, any solution of the equation

$$x[\tau_{i+1}] = f(\tau_i, x[\tau_i]) + u(\tau_i, x[\tau_i]) + v[\tau_i]$$
$$x[\tau_0] = x_0, \quad v[\tau_i] \in Q(\tau_i), \quad i = 0, \ldots, n-1$$

is called a polygonal line $x[\cdot] = x[\cdot, \tau_0, x_0, U, v[\cdot]]$. A motion $x[\cdot] = x[\cdot, \tau_0, x_0, U]$ of system (1.1), generated by strategy U, is any limit of the convergent sequence of polygonal lines $x_k[\cdot] = x[\cdot, \tau_0, x_0, U, v_k[\cdot]]$, i.e.

$$x[\tau_i] = \lim x_k[\tau_i], \quad k \to \infty$$

We formulate the following problem.

Problem 1. Find a strategy U_0 of the first player, satisfying the equality

$$\max_{x[\cdot]} \sigma(x[\vartheta, \tau_0, x_0, U_0]) = \min_U \max_{x[\cdot]} \sigma(x[\vartheta, \tau_0, x_0, U])$$

2. Let us derive the main results. For this purpose we introduce the following definitions. Suppose that some strategy U has been selected. We say that at instant τ_i the motion $x[\cdot] = x[\cdot, \tau_0, x_0, U]$ has passed into the state $s[\tau_i] = \{x[\tau_i], u[\tau_i]\}$ if there exists a sequence of polygonal lines $x_k[\cdot] = x[\cdot, \tau_0, x_0, U, v_k[\cdot]]$ converging to this motion and satisfying at instant τ_i the equality

$$u[\tau_i] = \lim u(\tau_i, x_k[\tau_i]), \quad k \to \infty$$

We shall say that the motion $x[\cdot] = x[\cdot, \tau_0, x_0, U]$ passes from state $s[\tau_i] = \{x[\tau_i], u[\tau_i]\}$ into state $s[\tau_{i+1}] = \{x[\tau_{i+1}], u[\tau_{i+1}]\}$ if we can find a sequence of polygonal lines $x_k[\cdot] = x[\cdot, \tau_0, x_0, U, v_k[\cdot]]$ converging to this motion and satisfying the relations

$$u[\tau_i] = \lim u(\tau_i, x_k[\tau_i]), \quad u[\tau_{i+1}] = \lim u(\tau_{i+1}, x_k[\tau_{i+1}]), \quad k \to \infty$$

By $u(\tau_i, s)$ we denote the control corresponding to state $\{\tau_i, s\} = \{\tau_i, x, u\}$. Any motion for which the equality

$$\rho = \sigma(x_0[\vartheta]) = \max_{x[\cdot]} \sigma(x[\vartheta, \tau_0, x_0, U]) \tag{2.1}$$

*Prikl.Matem.Mekhan.,Vol.47,No.3,pp.355-362,1983 is valid is called the maximizing motion $x_0[\cdot] = x_0[\cdot, \tau_0, x_0, U]$ for strategy U.

Let $S(\tau_i)$ $(i = 0, \ldots, n-1)$ be a collection of all states s through which the maximizing motion (2.1) passes at instant τ_i, i.e.

$$S(\tau_i) = \{s = \{x, u\},\ x = x_0[\tau_i, \tau_0, x_0, U],\\ u = u(\tau_i, s),\ \sigma(x_0[\vartheta, \tau_i, s, U]) = \rho\},\ i = 0, \ldots, n-1\\ S(\vartheta) = W = \{w, w = x_0[\vartheta, \tau_0, x_0, U]\} \tag{2.2}$$

By $S(\tau_{i+1}, \tau_i, s_*)$ we denote the set of all states through which the maximizing motion (2.1) passes at instant τ_{i+1} under the condition that this motion passed through state s_* at instant τ_i, i.e.

$$S(\tau_{i+1}, \tau_i, s_*) = \{s \in S(\tau_{i+1}),\ s = \{x_0[\tau_{i+1}],\ u[\tau_{i+1}]\},\\ u[\tau_{i+1}] = u(\tau_{i+1}, s),\ x_0[\tau_{i+1}] = x_0[\tau_{i+1}, \tau_i, s_*, U]\} \tag{2.3}$$

Let $\{\beta(A, \tau_{i+1}, \tau_i, s), A \subset S(\tau_{i+1}), s \in S(\tau_i), i = 0, \ldots, n-1\}$ be the system of regular probabilistic Borel measures concentrated on the sets $S(\tau_{i+1}, \tau_i, s)$ from (2.3) and integrable with respect to the $s \in S(\tau_i)$ from (2.2). For this system of measures we set

$$\psi(\tau_{n-1}, s_*) = \int_W \frac{\partial\sigma^*}{\partial x}(w)\,\beta(dw, \tau_n, \tau_{n-1}, s_*) \tag{2.4}$$

$$\psi(\tau_i, s_*) = \int_{S(\tau_{i+1})} \frac{\partial f^*}{\partial x}(\tau_{i+1}, x)\,\psi(\tau_{i+1}, s)\,\beta(ds, \tau_{i+1}, \tau_i, s_*)$$

$$(i = 0, \ldots, n-2)$$

$$\nu(A, \tau_{i+1}, \tau_i, s_*) = \beta(A, \tau_{i+1}, \tau_i, s_*)$$

$$\nu(A, \tau_j, \tau_i, s_*) = \int_{S(\tau_{i+1})} \nu(A, \tau_j, \tau_{i+1}, s)\,\beta(ds, \tau_{i+1}, \tau_i, s_*)$$

$$(j = i+2, \ldots, n-1)$$

$$\frac{\partial\sigma}{\partial x} = \left\{\frac{\partial\sigma}{\partial x_j}, j = 1, \ldots, m\right\}, \qquad \frac{\partial f}{\partial x} = \left\{\frac{\partial f_k}{\partial x_j}, \quad k, j = 1, \ldots, m\right\}$$

(G^* is the matrix adjoint to G).

Theorem 1. In order for strategy U to solve Problem 1 it is necessary that for this strategy there exist a system of measures $\{\beta(A, \tau_{i+1}, \tau_i, s), A \subset S(\tau_{i+1}), s \in S(\tau_i), i = 0, \ldots, n-1\}$ for which the equalities

$$\langle\psi(\tau_0, s_0)\cdot u(\tau_0, s_0)\rangle = \min_{u \in P(\tau_0)} \langle\psi(\tau_0, s_0)\cdot u\rangle \tag{2.5}$$

$$\int_{S(\tau_i)} \langle\psi(\tau_i, s)\cdot u(\tau_i, s)\rangle\,\nu(ds, \tau_i, \tau_0, s_0) = \int_{S(\tau_i)} \min_{u \in P(\tau_i)} \langle\psi(\tau_i, s)\cdot u\rangle\,\nu(ds, \tau_i, \tau_0, s_0) \quad (i = 1, \ldots, n-1)$$

are fulfilled, where $\langle p\cdot q\rangle$ is the scalar product of p and q, $\{\tau_i, s_*\} = \{\tau_i, x_*, u_*\}$, $u_* = u(\tau_i, s_*)$, the quantities $\psi(\tau_i, s)$, $\nu(\cdot, \tau_i, \tau_0, s_0)$ are prescribed in accord with (2.4).

We remark that the $\psi(\tau_i, s)$ in (2.4) are analogous to the adjoint functions from /2,5/. Let us consider a linear conflict-controlled system

$$x[\tau_{i+1}] = A(\tau_i)\,x[\tau_i] + u + v,\ x[\tau_0] = x_0 \tag{2.6}$$

for which the functions $\psi(\tau_i, s)$ are given by the formulas

$$\psi(\tau_{n-1}, s_*) = \int_W \frac{\partial\sigma^*}{\partial x}(w)\,\beta(dw, \tau_n, \tau_{n-1}, s_*) \tag{2.7}$$

$$\psi(\tau_i, s_*) = A^*(\tau_{i+1}) \int_{S(\tau_{i+1})} \psi(\tau_{i+1}, s)\,\beta(ds, \tau_{i+1}, \tau_i, s_*) \ (i = 0, \ldots, n-2)$$

Theorem 2. Let the discrete motions be described by the linear Eq.(2.6) and let the target function $\sigma(x)$ be convex. Then, in order for strategy U to solve Problem 1 it is necessary and sufficient that conditions (2.5), wherein the functions $\psi(\tau_i, s)$ have been defined by relations (2.7), be fulfilled for some system of measures $\{\beta(A, \tau_{i+1}, \tau_i, s),\ A \subset S(\tau_{i+1}), s \in S(\tau_i)\}$.

3. We derive auxiliary statements which will aid the proof of Theorem 1. Suppose that some strategy of the first player has been selected and that positive numbers α and ε have been prescribed. By $\{x_\alpha[\cdot] = x_\alpha[\cdot, \tau_0, x_0, U]\}$ we denote the collection of all α-maximizing motions for strategy U, i.e.

$$\sigma(x_\alpha[\vartheta]) \geqslant \rho - \alpha, \ \rho = \max_{x[\cdot]} \sigma(x[\vartheta, \tau_0, x_0, U]) \tag{3.1}$$

Let $S_\alpha(\tau_i)$ be the set of all possible states $s = \{x, u\}$ through which the α-maximizing motions (3.1) pass at instant τ_i, where

$$S_\alpha(\tau_n) = W_\alpha = \{w, w = x_\alpha[\vartheta, \tau_0, x_0, U], \sigma(w) \geqslant \rho - \alpha\}; S_\alpha(\tau_{i+1}, \tau_i, s_*)$$

is the set of all states through which the α-maximizing motions pass at instant τ_{i+1} under the condition that these motions passed through state s_* at instant τ_i; $\{\beta_\alpha(A, \tau_{i+1}, \tau_i, s_*), A \subset S_\alpha(\tau_{i+1}), s \in S(\tau_i), \ i = 0, \ldots, n-1\}$ is a system of regular probabilistic Borel measures concentrated on $S_\alpha(\tau_{i+1}, \tau_i, s_*)$ and integrable with respect to $s_* \in S(\tau_i)$.

We set

$$\psi_\beta(\tau_{n-1}, s_*) = \int\limits_{W_\alpha} \frac{\partial\sigma^*}{\partial x}(w)\, \beta_\alpha(dw, \tau_n, \tau_{n-1}, s_*) \tag{3.2}$$

$$\psi_\beta(\tau_i, s_*) = \int\limits_{S_\alpha(\tau_{i+1})} \frac{\partial f^*}{\partial x}(\tau_{i+1}, x)\, \psi_\beta(\tau_{i+1}, s)\, \beta_\alpha(ds, \tau_{i+1}, \tau_i, s_*)$$

$$(i = 0, \ldots, n-2)$$

$$\nu_\beta(A, \tau_{i+1}, \tau_i, s_*) = \beta_\alpha(A, \tau_{i+1}, \tau_i, s_*)$$

$$\nu_\beta(A, \tau_j, \tau_i, s_*) = \int\limits_{S_\alpha(\tau_{i+1})} \nu_\beta(A, \tau_j \tau_{i+1}, s)\, \beta_\alpha(ds, \tau_{i+1}, \tau_i, s_*)$$

$$(j = i+2, \ldots, n-1)$$

$$\begin{aligned}\varphi_\beta(\tau_i, s_*, z, u, \alpha, \varepsilon) = &\left\langle\left(\frac{\partial f^*}{\partial x}(\tau_i, x_*)\, \psi_\beta(\tau_i, s_*)\right)\cdot(z - x_*)\right\rangle + \\ &\varepsilon\langle\psi_\beta(\tau_i, s_*)\cdot(u - u(\tau_i, s_*))\rangle + \\ &\varepsilon\sum_{l=i+1}^{n-1}\int\limits_{S_\alpha(\tau_l)} \min_{u \in P(\tau_l)} \langle\psi_\beta(\tau_l, s)\cdot(u - u(\tau_l, s))\rangle\, \nu_\beta(ds, \tau_l, \tau_i, s_*)\end{aligned} \tag{3.3}$$

$$K = \max_{i, x, u} \max\left\{\left|\frac{\partial f}{\partial x}\right| + 1.2|u|\right\}, \quad M = \max_x\left|\frac{\partial\sigma}{\partial x}\right| \tag{3.4}$$

Here $|p|$ is the Euclidean norm of p, $x \in G$, where G is a compactum in $R^{(m)}$ containing all possible positions $x = x[\cdot]$ which result from (1.1) when $u \in P(\tau_i)$, $v \in Q(\tau_i)$ $(i = 0, \ldots, n-1)$. By analogy with the material in /1,4/ we define a strategy $U_{\alpha,\varepsilon} = u(\tau_i, s[\tau_i], z)$ with leader

$$x[\cdot, \tau_0, x_0, U], \ s[\tau_i] = \{x[\tau_i], u[\tau_i]\}$$

We say that the strategy $U_{\alpha,\varepsilon} = u(\tau_i, s[\tau_i], z)$ is a corrected strategy with leader $x[\cdot, \tau_0, x_0, U]$, $s[\tau_i] = \{x[\tau_i], u[\tau_i]\}$ if this strategy is specified by the following rule; for $s[\tau_i] \in S_\alpha(\tau_i)$ and $|x[\tau_i] - z| \leqslant \varepsilon K^i$ we set

$$u(\tau_i, s[\tau_i], z) = (1 - \varepsilon)\, u(\tau_i, s[\tau_i]) + \varepsilon u^*(\tau_i, s[\tau_i], z) \tag{3.5}$$

Here $u^* = u^*(\tau_i, s[\tau_i], z)$ is any control satisfying the relation

$$\begin{aligned}\max_\beta \varphi_\beta(\tau_i, s[\tau_i], z, u^*, \alpha, \varepsilon) = &\min_{u \in P(\tau_i)} \max_\beta \varphi_\beta(\tau_i, s[\tau_i], z, u, \alpha, \varepsilon) = \\ &\max_\beta \min_{u \in P(\tau_i)} \varphi_\beta(\tau_i, s[\tau_i], z, u, \alpha, \varepsilon) = d(\tau_i, s[\tau_i], z, \alpha, \varepsilon)\end{aligned} \tag{3.6}$$

where $\beta = \{\beta_\alpha(\cdot, \tau_{l+1}, \tau_l, s), \ l = i, \ldots, n-1\}$ while the function $\varphi_\beta(\tau_i, s, z, u, \alpha, \varepsilon)$ is defined in accord with (3.3), (3.2). If $s[\tau_i]$ does not belong to $S_\alpha(\tau_i)$ or if the inequality $|x[\tau_i] - z| > \varepsilon K^i$ is valid, then we set

$$u(\tau_i, s[\tau_i], z) = u(\tau_i, s[\tau_i]) \tag{3.7}$$

We note that by virtue of the continuity of $\partial\sigma/\partial x, \partial f/\partial x$ the compactness of sets $S_\alpha(\tau_{l+1}, \tau_l, s)$, as well as of (3.2) and (3.3) it follows that the maximum in (3.6) is achieved in the class of regular Borel measures $\beta = \{\beta_\alpha(A, \tau_{l+1}, \tau_l, s), A \subset S_\alpha(\tau_{l+1}), \ s \in S_\alpha(\tau_l), \ l = i, \ldots, n-1\}$ integrable with respect to $s \in S_\alpha(\tau_l)$. The operations of minimum and maximum in (3.6) can permute since $P(\tau_i)$ is a convex compactum, the function $\varphi_\beta(\tau_i, s, z, u, \alpha, \varepsilon)$ is linear in u, and $\beta_\alpha(\cdot, \tau_{i+1}, \tau_i, s)$ /6/. In addition, the motions $z[\cdot] = z[\cdot, \tau_i, s[\tau_i], z, U_{\alpha,\varepsilon}, v[\cdot]]$ are connnected with the motions of the leader $x[\cdot] = x[\cdot, \tau_i, s[\tau_i], U, v[\cdot]] = x[\cdot, \tau_0, x_0, U, v[\cdot]]$ by the equalities

$$z[\tau_{i+1}] = f(\tau_i, z[\tau_i]) + (1-\varepsilon)\, u(\tau_i, s[\tau_i]) + \varepsilon u^*(\tau_i, s[\tau_i], z[\tau_i]) + v[\tau_i]$$
$$x[\tau_{i+1}] = f(\tau_i, x[\tau_i]) + u(\tau_i, s[\tau_i]) + v[\tau_i] \tag{3.8}$$

if

$$s[\tau_i] = \{x[\tau_i], u[\tau_i]\} \in S_\alpha(\tau_i),\ |x[\tau_i] - z[\tau_i]| \leqslant \varepsilon K^i$$
$$z[\tau_{i+1}] = f(\tau_i, z[\tau_i]) + u(\tau_i, s[\tau_i]) + v[\tau_i] \tag{3.9}$$
$$x[\tau_{i+1}] = f(\tau_i, x[\tau_i]) + u(\tau_i, s[\tau_i]) + v[\tau_i]$$

if

$$s[\tau_i] \notin S_\alpha(\tau_i) \quad \text{or} \quad |x[\tau_i] - z[\tau_i]| > \varepsilon K^i$$

We observe that for the prescribed states $s[\tau_i]$ through which the motion $x[\cdot, \tau_0, x_0, U]$ passes, this motion is completely determined by the equations $v = v[\tau_i] \in Q(\tau_i)$ $(i = 0, \ldots, n-1)$.

Lemma 1. Suppose that the motion $x[\cdot] = x[\cdot, \tau_0, x_0, U]$ of (1.1) passed into the state $s[\tau_i] \in S_\alpha(\tau_i)$ at instant τ_i. Then for any positions z and realizations $v = v[\cdot]$ such that $|x[\tau_i] - z| \leqslant \varepsilon K^i$, $x[\vartheta] \in W_\alpha$, the strategy $U_{\alpha,\varepsilon}$ of (3.5) and (3.6) guarantees the estimate

$$\sigma(z[\vartheta]) \leqslant \rho + d(\tau_i, s[\tau_i], z, \alpha, \varepsilon) + o(\varepsilon) \tag{3.10}$$

where $z[\cdot] = z[\cdot, \tau_i, z, s[\tau_i], U_{\alpha,\varepsilon}, v[\cdot]]$ is the motion in (3.8) matched with the motion of the leader $x[\cdot, \tau_i, s[\tau_i], U, v[\cdot]]$, $o(\varepsilon)$ is a quantity of a higher order of smallness in comparison with ε, ρ, $d(\tau_i, s, z, \alpha, \varepsilon)$ are defined in accord with (3.1), (3.6).

Proof. At instant τ_{n-1}, for $s[\tau_{n-1}] \in S_\alpha(\tau_{n-1})$, $|x[\tau_{n-1}] - z| \leqslant \varepsilon K^{n-1}$, $x[\vartheta] \in W_\alpha$, from (3.5), (3.6), (3.8), (3.1) we have

$$\sigma(z[\vartheta]) \leqslant \rho + \max_{x[\cdot]} \Big[\Big\langle \frac{\partial \sigma^*}{\partial x}(x[\vartheta]) \cdot \Big(\frac{\partial f}{\partial x}(\tau_{n-1}, x[\tau_{n-1}])(z - x[\tau_{n-1}]) \Big) \Big\rangle +$$
$$\varepsilon \Big\langle \frac{\partial \sigma^*}{\partial x}(x[\vartheta]) \cdot (u^*[\tau_{n-1}] - u[\tau_{n-1}]) \Big\rangle \Big] + o(\varepsilon)$$
$$u^*[\tau_{n-1}] = u^*(\tau_{n-1}, s[\tau_{n-1}], z),\ u[\tau_{n-1}] = u(\tau_{n-1}, s[\tau_{n-1}])$$

Hence from (3.2), (3.3), (3.6) we obtain

$$\sigma(z[\vartheta]) \leqslant \rho + \max_\beta \Big[\Big\langle \Big(\frac{\partial f^*}{\partial x}(\tau_{n-1}, x[\tau_{n-1}])\, \psi(\tau_{n-1}, s[\tau_{n-1}]) \Big) \times (z - x[\tau_{n-1}]) \Big\rangle +$$
$$\varepsilon \langle \psi(\tau_{n-1}, s[\tau_{n-1}]) \cdot (u^*[\tau_{n-1}] - u[\tau_{n-1}]) \rangle \Big] + o(\varepsilon) = \rho + d(\tau_{n-1}, s[\tau_{n-1}], z, \alpha, \varepsilon) + o(\varepsilon)$$

Thus, inequality (3.10) has been proved for instant τ_{n-1}. We assume that (3.10) holds for $i = l+1$ and we obtain the required assertion for $i = l$. As a matter of fact, from (3.8), (3.5) it follows that if position z and state $s[\tau_l] \in S_\alpha(\tau_l)$ are connected by the relation $|x[\tau_l] - z| \leqslant \varepsilon K^l$, then the estimate $|x[\tau_{l+1}] - z[\tau_{l+1}]| \leqslant \varepsilon K^{l+1}$ will be fulfilled for the motions $z[\tau_{l+1}] = z[\tau_{l+1}, \tau_l, z, s[\tau_l], U_{\alpha,\varepsilon}, v[\cdot]]$, $x[\tau_{l+1}] = x[\tau_{l+1}, \tau_l, s[\tau_l], U, v[\cdot]]$ from (3.8). Then, according to (3.10), (3.2), (3.3), (3.6), (3.8), (3.1) we obtain

$$\sigma(z[\vartheta]) \leqslant \rho + \max_{\beta'\beta} \Big[\int_{S_\alpha(\tau_l)} \Big\{ \Big\langle \Big(\frac{\partial f^*}{\partial x}(\tau_{l+1}, x)\, \psi_\beta(\tau_{l+1}, s) \Big) \times$$
$$\Big(\frac{\partial f}{\partial x}(\tau_l, x[\tau_l])(z - x[\tau_l] + \varepsilon(u^*[\tau_l] - u[\tau_l])) \Big) \Big\rangle +$$
$$\varepsilon \min_{u \in P(\tau_{l+1})} \langle \psi_\beta(\tau_{l+1}, s) \cdot (u - u(\tau_{l+1}, s)) \rangle +$$
$$\varepsilon \sum_{k=l+2}^{n-1} \int_{S_\alpha(\tau_k)} \min_{u \in P(\tau_k)} \langle \psi(\tau_k, s_*) \cdot (u - u(\tau_k, s_*)) \rangle$$
$$\nu_\beta(ds_*, \tau_k, \tau_{l+1}, s) \Big\}\, \beta'_\alpha(ds, \tau_{l+1}, \tau_l, s[\tau_l]) \Big] + o(\varepsilon) =$$
$$\rho + \max_{\beta'\beta} \min_{u \in P(\tau_l)} \varphi_{\beta'\beta}(\tau_l, s[\tau_l], z, u, \alpha, \varepsilon) + o(\varepsilon) =$$
$$\rho + d(\tau_l, s[\tau_l], z, \alpha, \varepsilon) + o(\varepsilon)$$
$$(\beta' = \beta'_\alpha(\cdot, \tau_{l+1}, \tau_l, s), \quad \beta = \{\beta_\alpha(\cdot, \tau_{j+1}, \tau_j, s), \quad j = l+1, \ldots, n-1\}$$
$$\beta'\beta = \{\beta_\alpha(\cdot, \tau_{j+1}, \tau_j, s), \quad j = l, \ldots, n-1\})$$

The latter relation completes the Lemma's proof.

The next lemma follows immediately from Lemma 1 as well as from (3.7) and (3.9).

Lemma 2. One of the estimates: either

$$\sigma(z[\vartheta]) \leqslant \rho + d(\tau_0, s_0, x_0, \alpha, \varepsilon) + o(\varepsilon) \quad (3.11)$$

or

$$\sigma(z[\vartheta]) \leqslant \rho - \alpha + \varepsilon M K^n$$

is fulfilled for any motion $z[\cdot] = z[\cdot, \tau_0, x_0, U_{\alpha,\varepsilon}, v[\cdot]]$ of (3.8), (3.9), generated by an α, ε-corrected strategy with leader. Here $s_0 = \{x_0, u_0\}$, $u_0 = u(\tau_0, x_0)$ and the quantities $d(\tau_i, s, x, \alpha, \varepsilon)$, K, M are prescribed in accord with (3.6), (3.4).

Let $Z(\tau_i, \alpha, \varepsilon)$ be the set of all positions z for each of which we can find a motion $x[\cdot]_i = x[\cdot, \tau_0, x_0, U]$ satisfying the estimate $|x[\tau_i] - z| \leqslant \varepsilon K$. We introduce the set $Z_1(\tau_i, \alpha, \varepsilon)$; here $z \in Z_1(\tau_i, \alpha, \varepsilon)$ if for any motion $x[\cdot] = x[\cdot, \tau_0, x_0, U]$ satisfying the inequality $|x[\tau_i] - z| \leqslant \varepsilon K_i$ and any state $s[\tau_i] = \{x[\tau_i], u[\tau_i]\}$ through which this motion passed at instant τ_i, the inclusion $s[\tau_i] \in S_\alpha(\tau_i)$ is valid. Now we set

$$Z_2(\tau_i, \alpha, \varepsilon) = Z(\tau_i, \alpha, \varepsilon) \setminus Z_1(\tau_i, \alpha, \varepsilon), \quad Z_3(\tau_i, \alpha, \varepsilon) = R^{(m)} \setminus Z(\tau_i, \alpha, \varepsilon)$$

Here $A \setminus B$ is the difference of sets A and B, $R^{(m)}$ is an m-dimensional space. An α, ε-corrected positional strategy $U_{\alpha,\varepsilon} = u_{\alpha,\varepsilon}(\tau_i, z)$ is the function determined by the conditions

$$\begin{gathered} u_{\alpha,\varepsilon}(\tau_i, z) = (1 - \varepsilon)\, u(\tau_i, s_*) + \varepsilon u^*(\tau_i, s_*, z) \\ \varphi(\tau_i, s_*, z, u^*, \alpha, \varepsilon) = \min_s d(\tau_i, s, z, \alpha, \varepsilon) \\ z \in Z_1(\tau_i, \alpha, \varepsilon),\ s = \{x, u\},\ s_* = \{x_*, u_*\},\ |x_* - z| \leqslant \varepsilon K^i, \\ |x - z| \leqslant \varepsilon K^i \end{gathered} \quad (3.12)$$

(the functions $\varphi(\tau_i, s, z, u, \alpha, \varepsilon)$, $d(\tau_i, s, z, \alpha, \varepsilon)$ are defined by relations (3.3), (3.6))

$$\begin{gathered} u_{\alpha,\varepsilon}(\tau_i, z) = u(\tau_i, s_*) \\ s_* = \{x_*, u_*\} \notin S_\alpha(\tau_i),\quad z \in Z_2(\tau_i, \alpha, \varepsilon),\quad |x_* - z| \leqslant \varepsilon K^i \\ u_{\alpha,\varepsilon}(\tau_i, z) = u(\tau_i, z),\quad z \in Z_3(\tau_i, \alpha, \varepsilon) \end{gathered} \quad (3.13)$$

Comparing (3.12) and (3.13) with (3.6) and (3.7) from Lemmas 1 and 2, we obtain the following result.

Lemma 3. An α, ε-corrected positional strategy $U_{\alpha,\varepsilon}$ guarantees one of estimates (3.11) for any motion $z[\cdot] = z[\cdot, \tau_0, x_0, U_{\alpha,\varepsilon}]$.

Hence follows the next statement.

Lemma 4. In order for strategy U to solve Problem 1 it is necessary that for any $\alpha > 0$ a system of measures $\beta = \{\beta_\alpha(A, \tau_{i+1}, \tau_i, s), A \subset S_\alpha(\tau_{i+1}),\ s \in S_\alpha(\tau_i)\ (i = 0, \ldots, n-1)\}$ be found satisfying the conditions

$$\begin{gathered} \langle \psi_\beta(\tau_0, s_0) \cdot u(\tau_0, s_0) \rangle = \min_{u \in P(\tau_0)} \langle \psi_\beta(\tau_0, s_0) \cdot u \rangle \\ \int_{S_\alpha(\tau_i)} \langle \psi_\beta(\tau_i, s) \cdot u(\tau_i, s) \rangle\, \nu_\beta(ds, \tau_i, \tau_0, s_0) = \\ \int_{S_\alpha(\tau_i)} \min_{u \in P(\tau_i)} \langle \psi_\beta(\tau_i, s) \cdot u \rangle\, \nu_\beta(ds, \tau_i, \tau_0, s_0) \quad (i = 1, \ldots, n-1) \end{gathered} \quad (3.14)$$

Indeed, suppose that relations (3.14) are not fulfilled for some α. Then we can find $\varepsilon = \varepsilon(\alpha)$ such that for the positional strategy $U_{\alpha,\varepsilon}$ of (3.12), (3.13) we obtain, in accordance with Lemma 3.

$$\sigma(z[\vartheta, \tau_0, x_0, U_{\alpha,\varepsilon}]) < \rho = \max_{x[\cdot]} \sigma(x[\vartheta, \tau_0, x_0, U])$$

where $z[\cdot, \tau_0, x_0, U_{\alpha,\varepsilon}]$ is any motion generated by strategy $U_{\alpha,\varepsilon}$. Thus, strategy U is not a solution of Problem 1, which proves Lemma 4.

We now complete the proof of Theorem 1. We set $\alpha = \alpha(r) = 1/r$ $(r = 1, 2, \ldots)$. By Lemma 4 there exists a sequence of measures $\{\beta_\alpha(A, \tau_{i+1}, \tau_i, s), A \subset S_\alpha(\tau_{i+1}), s \in S_\alpha(\tau_i), i = 0, \ldots, n-1\}$, $\alpha = 1/r$, satisfying relations (3.14). From this sequence we can pick out a subsequence of measure $\{\beta_\alpha(A, \tau_{i+1}, \tau_i, s)\}$, which, on the strength of the inclusions $S(\tau_i) \subset S_\alpha(\tau_i)$, will weakly* converge to some system of measures $\{\beta(A, \tau_{i+1}, \tau_i, s), A \subset S(\tau_{i+1}), s \in S(\tau_i)\}$, concentrated on the sets $S(\tau_{i+1}, \tau_i, s), s \in S(\tau_i)$ of (2.2), (2.3). Here the functions $\psi_\beta(\tau_i, s)$, $\nu_\beta(\cdot, \tau_i, \tau_0, s_0)$ of (3.2) will converge to $\psi(\tau_i, s)$, $\nu(\cdot, \tau_i, \tau_0, s_0)$ of (2.4), which are defined by the system of measures $\{\beta(A, \tau_{i+1}, \tau_i, s), A \subset S(\tau_{i+1}), s \in S(\tau_i), i = 0, \ldots, n-1\}$. Consequently, equalities (2.5) follow from equalities (3.14), which proves Theorem 1.

4. We consider a conflict-controlled linear system (2.6) with the convex target function $\sigma(x)$. Theorem 2 can be proved by the scheme in /2/.

We present the proof of Theorem 2. The necessity of conditions (2.5) follows directly from Theorem 1. Let us assume that for some positional strategy U_* there exists a system of measures $\{\beta(A, \tau_{i+1}, \tau_i, s), A \subset S(\tau_{i+1}), s \in S(\tau_i), i = 0,\ldots, n-1\}$ for which relations (2.5) are fulfilled. We show that U_* solves Problem 1. Let U be any other positional strategy, $x[\cdot] = x[\cdot, \tau_0, x_0, U, v[\cdot]]$, $x_*[\cdot] = x_*[\cdot, \tau_0, x_0, U_*, v[\cdot]]$ be the motions generated by one and the same control $v[\cdot]$, where $x_*[\cdot, \tau_0, x_0, U_*, v[\cdot]]$ is the maximizing motion for strategy U_*. Then from (2.5)—(2.7) we obtain

$$\int\limits_{S(\tau_1)} \langle A^*(\tau_1)\psi(\tau_1, s_*)\cdot(x_*[\tau_1] - x[\tau_1])\rangle \beta(ds_*, \tau_1, \tau_0, s_0) =$$
$$\langle\psi(\tau_0, s_0)\cdot(u_*[\tau_0] - u[\tau_0])\rangle = \min_{u\in P(\tau_0)} \langle\psi(\tau_0, s_0)\cdot(u - u[\tau_0])\rangle \leqslant 0$$

where $u[\tau_0] = u(\tau_0, s_0)$ and the function $\psi(\tau_i, s)$ of (2.7) is specified in terms of the system of measures $\{\beta(A, \tau_{i+1}, \tau_i, s), A \subset S(\tau_{i+1}), s \in S(\tau_i)\}$ defined by strategy U_*. From the latter estimate we see that a set $D(\tau_1)$ of nonzero measure $\beta(\cdot, \tau_1, \tau_0, s_0)$ exists such that the inequality

$$\langle\psi(\tau_1, s_*[\tau_1])\cdot A(\tau_1)(x_*[\tau_1] - x[\tau_1])\rangle \leqslant 0$$

is fulfilled for all $s_*[\tau_1] \in D(\tau_1) \subset S(\tau_1)$. We assume further by induction that the estimate

$$\langle\psi(\tau_i, s_*[\tau_i])\cdot A(\tau_i)(x_*[\tau_i] - x[\tau_i])\rangle \leqslant 0 \tag{4.1}$$
$$s_*[\tau_i] \in D(\tau_i),\ \nu(D(\tau_i), \tau_i, \tau_0, s_0) > 0$$

is valid for instant τ_i. Let us show that a motion $x_*[\cdot, \tau_0, x_0, U, v[\cdot]]$ can be found, for which estimate (4.1) is preserved at instant τ_{i+1}. Indeed, from (2.5)—(2.7) and (4.1) we have

$$\int\limits_{S(\tau_{i+1})} \langle A^*(\tau_{i+1})\psi(\tau_{i+1}, s_*)\cdot(x_*[\tau_{i+1}] - x[\tau_{i+1}])\rangle \beta(ds_*, \tau_{i+1}, \tau_i, s_*[\tau_i]) =$$
$$\langle\psi(\tau_i, s_*[\tau_i])\cdot A(\tau_i)(x_*[\tau_i] - x[\tau_i])\rangle + \langle\psi(\tau_i, s_*[\tau_i])\cdot(u_*[\tau_i] - u[\tau_i])\rangle \leqslant$$
$$\min_{u\in P(\tau_i)} \langle\psi(\tau_i, s_*[\tau_i])\cdot(u - u[\tau_i])\rangle \leqslant 0$$

Therefore, a set $D(\tau_{i+1})$ $(\nu(D(\tau_{i+1}), \tau_{i+1}, \tau_0, s_0) > 0)$ exists such that the inequality

$$\langle\psi(\tau_{i+1}, s_*[\tau_{i+1}])\cdot A(\tau_{i+1})(x_*[\tau_{i+1}] - x[\tau_{i+1}])\rangle \leqslant 0$$

is fulfilled for all $s_*[\tau_{i+1}] \in D(\tau_{i+1})$. Hence, using (3.2) and the convexity of $\sigma(x)$, we finally obtain

$$\left\langle\frac{\partial\sigma^*}{\partial x}\cdot(x_*[\vartheta] - x[\vartheta])\right\rangle \leqslant 0$$
$$\sigma(x_*[\vartheta]) = \max_{x[\cdot]} \sigma(x[\vartheta, \tau_0, x_0, U_*]) \leqslant \max_{x[\cdot]} \sigma(x[\vartheta, \tau_0, x_0, U])$$

which proves Theorem 2.

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Translated by N.H.C.